axiomatic theorems one does not come to a full realization of what a proof is nor to a thorough understanding of just what the axioms do contain. The investigators into the foundations of geometry strive to be rid of all save the smallest possible number of axioms, to give proofs for all other propositions. This sometimes leads to great complication: for, the fewer our premises, the more elaborate must be our demonstration. From the pedagogic standpoint it is undoubtedly necessary to avoid the most difficult of these proofs and thus to posit more axioms than are of a strictly logical necessity.

For us the above theorem serves also as an excellent introduction to the theory of transformations and to the notations therein employed. A transformation, in this case a rigid motion, may conveniently be denoted by a letter placed in brackets. The expression "the point A and the direction a are carried by the motion $\mathbf{M}$ into A' and a' which in turn are carried back by $\mathbf{M}'$ into A and a" would be written in symbols as

$$A;\ a\ [\mathbf{M}]\ A';\ a'\ [\mathbf{M}']\ A;\ a.$$

The intermediate set of letters may be omitted and the expression becomes

$$A;\ a\ [\mathbf{M}]\ [\mathbf{M}']\ A;\ a \quad \text{or} \quad A;\ a\ [\mathbf{M}\,\mathbf{M}']\ A;\ a,$$

which is read "A and a are carried by the succession of $\mathbf{M}$ and $\mathbf{M}'$ into A and a. The inverse motion is denoted as a reciprocal, and the identical transformation is represented by unity,

$$\mathbf{M}', \text{ the inverse of } \mathbf{M}, \text{ is } \mathbf{M}^{-1};\ \mathbf{M}\mathbf{M}^{-1}=1.$$

Theorem 3. *The whole and the part of a segment cannot be congruent.*

Special Case. Let AB be any segment and B' any point of it, to show that

$$AB \text{ not} \equiv AB'.$$

Consider the direction $AB'B$. This remains fixed. The point A also remains fixed. By Theorem 1 all points must remain fixed. Hence B' cannot be carried into B. In a similar manner it can be shown that, if B' lies in AB produced,

$$AB \text{ not} \equiv AB'.$$

General Case. Let AB be any segment and $A'B'$ any segment which includes AB or is included by it. To show that the two cannot be congruent use may be made of the principles of order and of continuity. The proof will doubtless appear somewhat unfamiliar to those acquainted with only the ordinary presentations of geometry. It is nevertheless no more difficult than the proof of the theorems on limits which are included in all treatments of elementary geometry—that is, if those proofs are correctly given instead of wrongly as is usually the case. By Theorem 2, $A'B'$ must be congruent to AB if AB is congruent to

$A'B'$. Consequently it may be assumed without loss of generality that $A'B'$ is included by AB. Suppose **M** is the transformation which carries AB into $A'B'$. Consider the points $A''B''$ into which $A'B'$ are carried, the points into which these points $A''B''$ are carried, and so on. We have

$$AB\,[\mathbf{M}]\,A'B',\quad A'B'\,[\mathbf{M}]\,A''B'',\quad A''B''\,[\mathbf{M}]\,A'''B''',\ldots\ldots\ldots$$

or

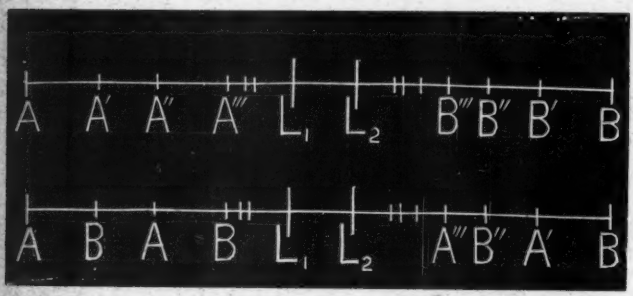
$$AB\,A'B'\,A''B''\ldots\ldots\ldots\,[\mathbf{M}]\,A'B'\,A''B''\,A'''B'''\ldots\ldots\ldots$$

According to Axiom V_2, as explained in the footnote to Theorem 2, lines are transformed into lines in such a manner as to preserve the property of order. Hence

$$AA'B,\quad A'A''B',\quad A''A'''B'',\ldots\ldots\ldots$$

have the same order upon the line AB. The order opposite to this is shared in common by the series of points

$$BB'A,\quad B'B''A',\quad B''B'''A'',\ldots\ldots$$



As shown in Fig. 1, two different cases may occur. In the first of them the points

$$A,\ A',\ A'',\ A''',\ldots\ldots\ldots$$

move constantly in the same direction and the points

$$B,\ B',\ B'',\ B''',\ldots\ldots\ldots$$

in the opposite direction. In the second case the points

$$A,\ B',\ A'',\ B''',\ldots\ldots\ldots$$

move in one direction and the points

$$B,\ A',\ B'',\ A''',\ldots\ldots\ldots$$

in the direction opposite to them. In either case on examining the order of $AA'B'$, $A'A''B''$, $A''A'''B''$, etc., it is seen that the points of the first set lie before these of the second set. We have, then, two series of points continually approaching but never passing one another. Under these circumstances it is geometrically evident (and it will be proved in the next theorem) that the points of each set must approach a limiting point which may be the same for both sets. Let L_1 be the limit of the first set in the first case. L_1 is not changed in position by the motion **M**. On the proof of this turns the demonstration of the theorem. L_1 cannot move to the left; for if it did it would fall in one of the

segments $A^{(n)}A^{(n+1)}$ and we know that the points which make up this segment have come from the segment $A^{(n-1)}A^{(n)}$. L_1 cannot move to the right; for if it did any point X situated between the original and the new positions of L_1 would have to come from some point to the left of L, that is, from some segment $A^{(n)}A^{(n+1)}$ all of whose points we know go into the segment $A^{(n+1)}A^{(n+2)}$. Hence in the first case L_1 remains at rest. Therefore L_1A [**M**] L_1A', which is contradictory to what has been shown in the special case of this theorem. As we have deduced by logical steps an impossibility from reasoning on the assumption that AB may be carried into $A'B'$, it is evident that that assumption must have been wrong. To treat the second case it is best not to use the transformation **M**, but **M** followed by **M**. Then the points with an odd accent may be neglected.

$$ABA''B''\ldots\ldots[\mathbf{M}]\,[\mathbf{M}]\,A''B''A''''B''''\ldots\ldots\ldots\ldots$$

For the transformation **M M** this case offers the same difficulty as the former did for **M**. Therefore the second supposition is also impossible and the theorem that a whole and a part of a segment cannot be congruent is proved in its entirety.

Theorem 4. *Given a series of collinear points A, A', A'', ... the members of which move continually in the same direction without, however, passing a certain point B. There must exist a limiting point L to this series and this limiting point does not lie beyond B.**

Consider any point X of the segment AB. Either there are members of the series between X and B or there are not. Separate the points of the segment into two classes corresponding to these two assumptions. Let X be any point of the first class and X' any point of the second. There is some point $A^{(n)}$ of the series between X and B. Hence X comes before $A^{(n)}$; but X' cannot come before $A^{(n)}$. Hence X comes before X' and every point of the first class precedes every point of the second class. By Axiom IV, there exists a point L such that any point before L belongs to the first class and any point after L belongs to the second. To show that L is the limiting point of the series it is necessary to have some perfectly definite definition of limiting point. The idea of a limit is generally connected with that of nearness. The statement of the definition is expressed in terms of distance. For us at the present stage of development this is impossible because no measure of distance has been adopted and indeed it has not yet been stated how distance can be measured. A more general definition of the limiting point, one that is independent of distance, must be given. If L be the limiting point of a series of points situated on a line, and if we choose from the line any segment which contains in its interior the point L, then from a certain point on all the points of the series will lie in this segment. Conversely, *if, when an arbitrary segment surrounding the point L has been chosen, all the points of the series from some point on lie in this segment, then the point L is said to be the limiting point of the series.* In case the points of the series all lie on the same side of the point L the segment does not surround L but may terminate in L and lie on that side of L on which the series lies. Furthermore,

*Figure 1, or one quite similar to it, may serve the purpose.

when the points of the series always move in the same direction on the line it is sufficient to show that one point of the series lies in the arbitrarily chosen segment: for then all the rest must lie in it. It is this last, simplest state of affairs which obtains in the case in hand. Let L be the point which separates the class of points X from the class of points X' as stated above. Let LM be any segment lying on that side of the point L on which the points $A, A', A'', \ldots\ldots$ lie. The point M belongs to the first class and hence there must be some point $A^{(n)}$ of the series between M and B. On the other hand there are no such points between L and B. Hence the point $A^{(n)}$ lies in the segment LM. By the definition given above the point L is therefore the limiting point of the series $A, A', A'', \ldots\ldots$ and the theorem is proved.

The proofs of Theorems 3 and 4 required the use of the principle of continuity. When the demonstrations are critically examined it will be seen that three essential ideas are involved: first, the separation of the points of a segment into two classes such that any point of one lies bofore any point of the other; second, the application of the axiom of continuity to establish the existence of a certain point L; third, the proof that this point has the desired properties. This sort of demonstration occurs so frequently in any careful development of geometry that it is well for the reader to become thoroughly familiar with it at an early stage. A little practice will render it both clear and easy. There is no necessity of tacitly assuming that a circle is a continuous curve—the fact may be proved. In like manner it may be proved that a segment may be divided into any desired number of parts. In Euclidean geometry such a proof is scarcely necessary, for a construction may be found to accomplish that division. In spherical geometry, however, no such construction can be found and recourse must be had to the principle of continuity to show that such a thing as the third part of a segment exists.

Theorem 5. *A segment is congruent to itself reversed in direction and in this congruence one point of the segment remains fixed so that the portions of the segment on the opposite sides of the point merely change places.*

Let AB be a segment. To prove the first part of the theorem it is only necessary to note that the point A may be applied to the point B and the direction AB to the direction BA (Axiom V_3). Then B will fall upon A: for otherwise the part and the whole would be congruent. To prove the second statement of the theorem the positions of corresponding points of the segments AB and BA may be considered. The details of this demonstration are left to the reader. The entire theorem may be proved at once by having recourse to the principle of continuity. Let O be any point of AB. If O be held fixed and the direction OB brought into coincidence with the direction OA, the point B' into which the point B is carried lies either between A and B or at A or beyond A. Divide the points of the segment AB into two classes such that to the first class belong those points O for which B' lies between A and B. By the application of Axiom IV we may show that there exists a point L which remains fixed and which has the property that $LA \equiv LB$. The details of the proof are left to the reader.

ON THE GENERALIZATION AND EXTENSION OF SYLOW'S THEOREM.

By PROFESSOR G. A. MILLER, Stanford University.

Cauchy proved that every prime number which divides the order of a group (G) is itself the order of at least one subgroup of G. Let p represent any prime which divides the order of G and suppose that α has been so chosen that p^α is the highest power of p which is the order of a subgroup of G. Such a subgroup may be denoted by P_α, and we may suppose that P_α contains only one subgroup (P_β) of order p^β and of a particular type. This condition is always fulfilled when $\beta=\alpha$, but, in many cases, it may also be satisfied for smaller values of $\beta>0$.

We proceed to prove that the number of the subgroups in G which are of the same type as P_β is of the form $1+kp$. If one of these subgroups would transform another into itself the two would generate a group ($P_{\beta'}$) of order $p^{\beta'}$, $\beta'>\beta$. Hence the number of the subgroups of order p^α is of the form $1+kp$, and every operator of order p^γ which transforms one of these subgroups into itself must be in this subgroup. Since no subgroup of order p^β could transform each one of the $1+kp$ subgroups of order p^α into another, it follows that every subgroup of order p^β is found in $1+mp$ subgroups of order p^α.* In particular, $P_{\beta'}$ must be contained in a subgroup of order p^α. This is impossible if all the subgroups of order p^α are conjugate since $P_{\beta'}$ contains at least two subgroups of the same type as P_β while P_α contains only one such subgroup.

That the $1+kp$ subgroups of order p^α form a single conjugate set may be seen as follows: If they did not form a single set, they could be divided into more than one set such that each set would include only those which are conjugate under G. Since P_α (or any subgroup of P_α which is not contained in another group of order p^α) transforms each of the other subgroups of order p^α into a different one, it follows that the set which includes P_α must include $1+k'p$ subgroups while each other set must include a multiple of p subgroups. Since P_α is any of the subgroups of order p^α this result is impossible. Hence all the subgroups of order p^α form a single conjugate set and the number of the subgroups of order p^β which are of the same type as P_β is of the form $1+kp$. The latter constitute a single conjugate set since the subgroups of order p^α form a single congugate set.

Since p^α is the order of a subgroup of G it must divide the order of G. That no higher power of p divides this order follows from the fact that the order is $p^\alpha h(1+kp)$, where $p^\alpha h$ is the order of the largest subgroup of G which transforms P_α into itself. If h were divisible by p, G would contain a subgroup of order $p^{\alpha+1}$. Hence we have the results:

*Several of the arguments of this proof depend upon the fact that any group of order p^α transforms operators and subgroups in sets of p^β, where β may be 0.

(1) *The number of subgroups which are of the same type as* P_β *is of the form* $1+kp$.

(2) *All of these subgroups form a single conjugate set.*

(3) *The order of* G *is of the form* $p^\beta h_1(1+kp)$, *where* $p^\beta h_1$ *is the order of the largest subgroup of* G *which transforms* P_β *into itself.*

By letting $\beta=\alpha$ in these results we have Sylow's theorem. When $\beta=\alpha$ the factor h_1 is not divisible by p, while it is divisible by p for all other values of β.

If G is represented as a substitution group it is only necessary that P_α does not contain two similar subgroups of order p^β. It may be observed that characteristic subgroups of P_α may be conjugate under G. For instance, in the simple group of order 168 all the subgroups of order 2 are conjugate while the subgroups of order 8 contain a characteristic subgroup of order 2. If P_α contains just m sets of conjugate subgroups of a given order, G cannot contain a larger number of such sets but it may contain a smaller number. In particular, all the subgroups of order p^α in G are conjugate since P_α contains only one subgroup of this order.

The preceding relates to generalizations of Sylow's theorem. The main extension of this fundamental theorem is due to Frobenius and affirms that the number of subgroups of order p^δ, $\delta \leqq \alpha$, is of the form $1+kp$, but that these subgroups do not necessarily form a single set of conjugates except when $\delta=\alpha$. It is clear that every subgroup of order p^δ which is transformed into itself by P_α must be contained in P_α. Hence P_α transforms all the subgroups of order p^δ which are not contained in it in sets each containing p^ε ($\varepsilon>0$) subgroups. That is, if the number of subgroups of order p^δ in P_α is of the form $1+kp$, the number of those in G must be of the same form.

It remains to prove that the number of invariant subgroups of order p^δ in P_α is of the form $1+kp$. Let P_δ represent any one of these invariant subgroups. All the others have some largest subgroup in common with P_δ. The common subgroup is clearly invariant under P_α. We proceed to prove that the number of those subgroups of order p^δ (excluding P_δ) which have a particular subgroup in common with P_δ is always a multiple of p. If the common invariant subgroup is not the identity, the number of these is equal to the number of subgroups (less one) of a certain order in the corresponding quotient group. As the theorem may be supposed true for for all groups whose order is less than p^α it may be assumed true for each one of these quotient groups. That is, the number of invariant subgroups of order p which have a particular subgroup (not the identity) in common with P_δ and are different from P_δ is a multiple of p.

It remains only to consider the case when the largest common subgroup is the identity. In this case P_α contains one or more direct products of P_δ and some other group of order p^δ. In such a direct product $[P_\delta, P_\delta']$ all the subgroups of order p^δ which have only the identity in common with P_δ can be obtained by establishing a β, 1 isomorphism between P_δ' and some subgroup of P_δ. Moreover, all the groups which can be constructed in this manner are in this direct product. For any particular subgroup of P_δ whose order exceeds p the number

of such isomorphisms is a multiple of p, since the order of the group of isomorphisms of any group of order p^γ $(\gamma>0)$ is divisible by p. Hence there are always a multiple of p such subgroups for every possible value of $\beta<p^{\delta-1}$.

When $\beta=p^\delta$ there is clearly only one group, viz, P_δ'. When $\beta=p^{\delta-1}$ there are $p-1$ such groups for every subgroup of order $p^{\delta-1}$ in P_δ' and any particular subgroup of order p in P_δ. Since $\delta<\alpha$, we may assume that the required theorem is true with respect to P_δ and P_δ'. Hence the number of the required subgroups for $\beta=p^{\delta-1}$ and p^δ is of the form $(n_1p+1)(n_2p+1)(p-1)+1$. That is, the number of subgroups of order p^δ which have only identity in common with P_δ and are contained in the direct product $[P_\delta, P_\delta']$ is also a multiple of p. All of these subgroups are simply isomorphic with P_δ'. This completes the proof that P_α contains just $1+kp$ subgroups of order p^δ.

In some cases it is easy to extend this extension of Sylow's theorem. For instance, we may readily see that the number of subgroups of order $p^{\alpha-1}$ in P_α is of the form $\frac{p^m-1}{p-1}$.* To see this interesting fact we may observe that any subgroup of order $p^{\alpha-1}$ has just $p^{\alpha-2}$ operators in common with any other subgroup of this order. These common operators form an invariant subgroup of P_α, which includes its commutator subgroup since the corresponding quotient group is abelian. It also includes the pth power of every operator of P_α since this quotient group is of type (1, 1). Hence the operators which are common to all the subgroups of order $p^{\alpha-1}$ form an invariant subgroup including the pth power of every operator of P_α as well as its commutator subgroup. The corresponding quotient group is therefore abelian and of the type (1, 1, 1, ...). Since the subgroups of order p^{m-1} in this quotient group correspond to groups of order $p^{\alpha-1}$ in P_α and since the former number is equal to the number of sub-groups of order p in the quotient group of order p^m, the theorem is proved.

Another interesting extension is the fact that the number of subgroups of order p in P_α is of the form $1+p+kp^2$ whenever there is more than one such subgroup and p is odd. This extension is clearly included in the following theorem:

All the operators of order p in P_α which have not more than p conjugates constitute (with the identity) a subgroup. If all of these operators are invariant the theorem is evident. If this is not the case we may associate a subgroup of order $p^{\alpha-1}$ with each of the non-invariant operators of order p. The common operators of all these subgroups form an invariant subgroup which includes the commutator subgroup of P_α and is composed of the operators which are commutative with each of the operators of the group generated by the given operators of order p. The required theorem follows directly from this property if we bear in mind the well known equations

$$\left.\begin{array}{l} t^{-n}st^n=s_1^{\,s} \\ (st^{-1})^n=s_1^{\,[n(n-1)]/2t-n} \end{array}\right\} \text{whenever } s_1 \text{ and } s \text{ are commutative.}$$

A somewhat different way of stating this theorem is as follows: If a

*Bauer, Nouvelles Annales de Mathematiques, 1900, Vol. 19, p. 508.

number of operators of order $p>2$ generate a group under which each one of these operators has no more than p conjugates then the entire group contains no operator whose order exceeds p. It should however be observed that while this condition is sufficient it is by no means necessary to insure the fact that the resulting group contains no operator whose order exceeds p.

It seems well to add that the greater part of this paper aims to present known results in a connected form. The theorem relating to the number of subgroups of order p is, however, supposed to be new and evidently admits of further generalization. The method of proving Frobenius' extension of Sylow's theorem is also supposed to be new.

THE THEORY OF OPTICAL SQUARES.

By PROFESSOR ARNOLD EMCH, The University of Colorado.

1. Optical squares* as they are used in surveying owe their usefulness to the fact that the incident rays from an object include constant angles with the corresponding final reflected rays. The proof for this constancy is a simple application of the geometric proposition that the sum of the angles of any triangle is equal to two right angles.

The ordinary optical square is an entirely special case of a multiple "square" whose geometrical theory and its most important application I propose to present in this paper.

2. This theory results from a close investigation of Poncelet's famous problem: *To construct a rectilinear polygon whose vertices shall lie on given straight lines (one on each) and whose sides shall pass through given points (one through each).*

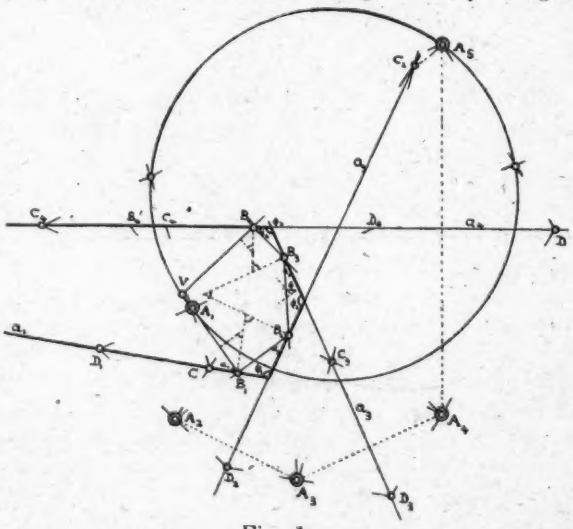


Fig. 1.

*For a description of optical squares (angular mirrors) see G. F. Barker's Physics, Advanced Course, pp. 410-414, or any good text-book on surveying.

†Traite des proprietes projectives des figures (1822), p. 345, or 2nd ed., tome I, p. 281. See also Cremona: Elements of projective geometry, p. 184.

Although the solution of this problem is well known it is convenient for a clearer understanding of what follows to give a brief outline of it.

Let $A_1, A_2, A_3, \ldots, A_n$ be the given points and $a_1, a_2, a_3, \ldots, a_n$ the given lines. Through A_1 draw any line v_1 cutting a_1 in B_1 and a_n in B'_n. Through A_2 and B_1 draw a line v_2 cutting a_2 in B_2, through A_3 and B_2 a line v_3 cutting a_3 in B_3, and so forth; finally through A_n and B_{n-1} a line v_n cutting a_n in B_n. In Fig. 1 this construction is shown in the case of $n=4$.

Repeating this construction with any number of initial rays v_1 through A_1, a number of trial polygons is obtained whose n sides pass through the given points and whose vertices, with the exception of the points of intersection of the pairs of rays v_1 and v_n, lie on the given sides. The pencils of rays through the different points formed by the sides of these polygons are all projective, as it is evident from the construction. Hence the two pencils through A_1 and A_n formed by the rays v_1 and v_n respectively produce a conic through A_1 and A_n. This conic cuts the line a_n either in two real or two imaginary points, or in a special case in two coincident points. Starting from these points by drawing lines as required by Poncelet's problem the required polygons are obtained. To actually get those closing points on a_n we have to draw three trial-polygons. Let the initial sides (v_1) of these three polygons cut a_n in B_n', C_n', D_n' and the three nth sides cut a_n in B_n, C_n, D_n. The six points B_n, C_n, D_n and B_n', C_n', D_n' determine the two coincident projective ranges on a_n and their self-corresponding pair which is found by a well-known construction gives the required points.

2. A problem which depends on similar considerations may be stated in the form: To construct a polygon whose sides $s_1, s_2, s_3, \ldots, s_n$ pass through given points $A_1, A_2, A_3, \ldots, A_n$ respectively, and whose vertices V_1 (intersection of s_1 and s_2), $V_2(s_2, s_3)$, $V_3(s_3, s_4)$, ..., $V_{n-1}(s_{n-1}, s_n)$ lie on given straight lines $a_1, a_2, a_3, \ldots, a_{n-1}$, respectively. It is further required that the rays on sides s_1 and s_n include a given angle.

To solve this problem we first construct a conic through A_1 and A_n, exactly as in Poncelet's problem. Then through A_1 and A_n draw a circle whose points subtend the given angle over A_1A_n as a chord. Where this circle cuts the conic are the two points where the rays s_1 and s_n of the required polygon (generally two polygons) intersect each other. If the conic is itself a circle, then there is generally no solution of the problem. If the conic is identical with the circle through A_1 and A_n subtending the given angle, then there is an infinite number of solutions. At it will be seen in the following sections, also this problem bears on the theory of optical squares.

4. Cremona in his Elements of Projective Geometry, page 199, states an optical problem which, as it will be seen, is an application of the foregoing proposition.

A ray of light emanating from a given point A_1 is reflected from n given straight lines in succession; to determine the original direction which the ray must have, in order that this may make with its direction, after the last reflection, a given angle.

Physically, this is equivalent to n reflecting mirrors, all perpendicular

to a plane which is parallel to the rays of light and whose intersections with the mirrors are the lines $a_1, a_2, a_3, \ldots, a_n$.

For the sake of generality we make the hypothesis that these mirrors may be indefinitely extended and that reflections may occur along their whole extent and always in such a direction that the ray reflected from the mirror a_i will strike the mirror a_{i+1}. In other words, we assume that a ray of light may be indefinitely produced beyond its physical boundaries. This hypothesis is geometrically possible and includes the real physical cases.

Designate in Fig. 1 the reflecting lines or mirrors by $a_1, a_2, a_3, \ldots, a_n$. Through A_1 draw any ray v_1 striking a_1 at B_1. The reflected ray v_2 passes through the point A_2 which is symmetrical to A_1 with respect to a_1. The ray v_2 strikes a_2 at B_2 and its reflected ray v_3 passes through A_3 which is symmetrical to A_2 with respect to a_2 as an axis, and so forth. The ray v_{n+1} reflected from the last line a_n at B_n passes through A_{n+1} which is symmetrical to A_n with respect to a_n as an axis. Let the rays v_1 and v_{n+1} intersect at V. We have now a closed polygon whose sides $v_1, v_2, v_3, \ldots, v_{n+1}$ pass through the fixed points points $A_1, A_2, A_3, \ldots, A_{n+1}$ and whose vertices $B_1, B_2, B_3, \ldots, B_n$ except V, lie on the fixed lines $a_1, a_2, a_3, \ldots, a_n$. Hence, when v_1 turns about A_1, V will describe a conic and the problem is reduced to the one explained under (3).

5. This is as far as Poncelet's and Cremona's discussions of these problems go. It will be seen, however, that an investigation of the angular relations between the rays of light forming the polygon $v_1, v_2, \ldots, v_{n+1}$ brings to light some interesting facts.

Let α_1 be the angle of incidence of v_1 on a_1, α_2 the angle of incidence of v_2 on a_2, α_3 of v_3 on a_3, and so forth; $\phi_1, \phi_2, \phi_3, \ldots$ the angles which a_1 and a_2, a_2 and a_3, a_3 and a_4, ..., include. The angles $\phi_1, \phi_2, \phi_3 \ldots$, according to the physical requirements of the problem must be chosen in such a manner that always

$$\alpha_i + \alpha_{i+1} + \phi_i = \pi.$$

From the figure we now derive the following series

$$\begin{aligned}
\alpha_1 &= \alpha_1, \\
\alpha_2 &= \pi - \alpha_1 - \phi_1, \\
\alpha_3 &= \alpha_1 + \phi_1 - \phi_2, \\
\alpha_4 &= \pi - \alpha_1 - \phi_1 + \phi_2 - \phi_3, \\
\alpha_5 &= \alpha_1 + \phi_1 - \phi_2 + \phi_3 - \phi_4, \\
\alpha_6 &= \pi - \alpha_1 - \phi_1 + \phi_2 - \phi_3 + \phi_4 - \phi_5, \\
&\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\
\alpha_{2\mu} &= \pi - \alpha_1 - \phi_1 + \phi_2 - \phi_3 + \ldots\ldots - \phi_{2\mu-1}, \\
\alpha_{2\mu+1} &= \alpha_1 + \phi_1 - \phi_2 + \phi_3 - \ldots\ldots - \phi_{2\mu}.
\end{aligned}$$

If there are n reflecting lines, then the number of angles α is also n, and the number of angles ϕ is $n-1$. To find the deviation of the first ray v_1 from the final reflected ray v_{n+1}, drop a perpendicular from any point to the lines v_1 and

v_{n+1}. Erect perpendiculars to a_1, a_2, a_3, ..., a_n at B_1, B_2, B_3, ..., B_n respectively. Then, the first and last perpendicular deviate from each other by the angle

$$a_1+(n-2)\pi+a_n-(\phi_1+\phi_2+\phi_3+\ldots\ldots+\phi_{n-1}),$$

consequently also the rays v_1 and v_{n+1} by the same angle

$$\phi=a_1+a_n-(\phi_1+\phi_2+\phi_3+\ldots\ldots+\phi_{n-1})+(n-2)\pi.$$

If n is odd, $n=2\mu+1$,

$$\begin{aligned}\phi&=a_1+a_1+\phi_1-\phi_2+\phi_3-\ldots\ldots-\phi_{2\mu}-\phi_1-\phi_2-\phi_3-\ldots\ldots-\phi_{2\mu}+(n-2)\pi,\\&=(n-2)\pi+2a_1-2(\phi_2+\phi_4+\ldots\ldots+\phi_{2\mu}).\end{aligned}$$

In this case the angle between the original incident ray and the final reflected ray depends on the angles ϕ and the original angle of incidence a_1. Their point of intersection V describes a conic and there are two positions of V, real or imaginary, for which the incident and reflected ray make a given angle.

If n is even, $n=2\mu$,

$$\begin{aligned}\phi&=a_1+\pi-a_1-\phi_1+\phi_2-\phi_3+\ldots\ldots-\phi_{2\mu-1}-\phi_1-\phi_2-\phi_3-\ldots\ldots-\phi_{2\mu-1}+(n-2)\,\pi,\\&=(n-1)\pi-2(\phi_1+\phi_3+\ldots\ldots+\phi_{2\mu-1}).\end{aligned}$$

From this value of ϕ it is seen that in case of an even number of reflecting sides,

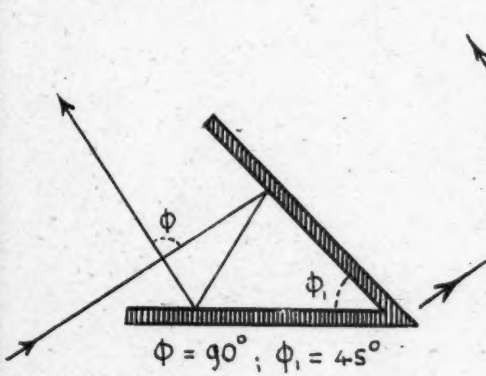


Fig. 2.

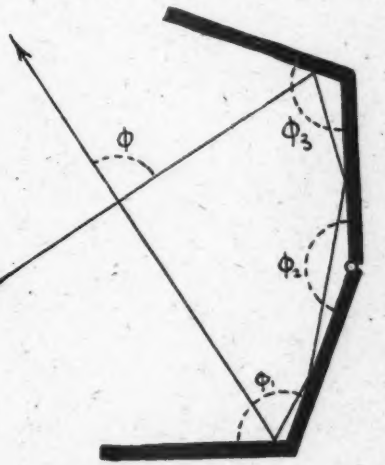


Fig. 3.

V describes a circle and the angle ϕ is constant. Cremona's optical problem admits either of no solution or of an infinite number of solutions and the angle ϕ does not depend on the angles ϕ_i of even indices.

We have therefore proved the following proposition:

If rays of light emanate from a fixed source which, in succession, are reflected on n straight lines (mirrors) then the final reflected rays cut the corresponding original rays in points of a conic which is not or is a circle according as n is odd or even. In the first case (n odd) there are two places on the conic where the original and final rays make a given angle. In the second case (n even) there are no such places on the circle, or else an infinite number. In this case (n even) the angle ϕ only depends on the angles between succeeding reflecting lines, whose orders in this succession are odd.

6. APPLICATIONS. Let $n=2$, then $\phi=\pi-2\phi_1$. To make $\phi=\frac{1}{2}\pi$, we must choose $\phi_1=45°$. This is illustrated in Fig. 2, and is practically applied in Adams' *Angle Mirror** or *Optical Square*.

For $n=4$, $\phi=3\pi-2(\phi_1+\phi_3)$.

To make $\phi=\frac{1}{2}\pi$, $\phi_1+\phi_3$ must be made equal to $\frac{5}{4}\pi$. Under this condition ϕ_1 and ϕ_3 may vary separately. The condition is also satisfied by taking $\phi_1=\phi_2=\frac{5}{8}\pi=112° 30'$, and this is illustrated in Fig. 3.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Solutions of Problem 188 have been received from J. E. Sanders, Hackney, Ohio, and Charles A. Carpenter, Student in Adelphi Academy, Brooklyn, N. Y.

189. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Solve $3x-y=xy-x^2$, $\sqrt[3]{y}=\sqrt{\frac{2y-x}{2}}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

From the first equation, $y=\frac{x^2+3x}{x+1}$.

$$\therefore \sqrt[3]{\frac{x^2+3x}{x+1}}=\sqrt{\frac{x^2+5x}{2(x+1)}}.$$

$$\therefore x^2(x^4+7x^3+19x^2+5x-72)=0.$$

$$\therefore x=0,\ 1.442,\ -3.65,\ -2.396\pm 2.829\sqrt{-1};$$
$$y=0,\ 2.623,\ -.895,\ 7.583\mp .423\sqrt{-1}.$$

*Invented by George Adams in London in the middle of the 18th century and first described by his son in 1791.

GEOMETRY.

Solutions of Problem 210 have been received from J. E. Sanders, Hackney, O.; Charles E. Barrett, Anchorage, Ky., and Charles A. Carpenter, Student in Adelphi Academy, Brooklyn, N. Y.

Solutions of Problem 211 have been received from J. E. Sanders, Hackney, O., and John D. Cutter, Student in Adelphi College, Brooklyn, N. Y.

174. Proposed by B. F. FINKEL, A. M., 204 St. Marks Square, Philadelphia, Pa.

Given two triangles ABC and $A'B'C'$ lying in the same plane. The side $B'C'$ cuts the sides AC, BC, and AB in the points I, H, and G, respectively; the side $A'B'$ cuts the sides AC, BC, and AB in D, F, and E, rrspectively; and $A'C'$ cuts AC, BC, and AB in M, L, and K, respectively. Prove that

$$(DA'.EA'.A'F)(GB'.HB'.B'I)(MC'.LC'.C'K)$$
$$=-(KA'.A'L.A'M)(FB'.B'E.B'D)(IC'.C'H.C'G).$$

Solution by G. W. GREENWOOD, A. M. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Since AC is a transversal cutting the sides $A'B'$, $B'C'$, $C'A'$ of the triangle $A'B'C'$ in the points D, I, M, respectively, we have, by the theorem of Menelaus,

$$A'D.B'I.C'M=-B'D.C'I.A'M.$$

Writing down the corresponding results for the other transversals, we get, by multiplying, the result required.

Also solved by G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.

213. Proposed by H. F. MacNEISH, University High School, Chicago, Ill.

Construct an equilateral triangle which shall have its vertices in three given parallel lines.

Remark by G. I. HOPKINS, Manchester, N. H.

The solution follows easily from the solution of Geometry Problem number 156 in the MONTHLY for November, 1901.

Independent solutions have been received from R. A. Wells, Bellevue, Neb.; G. W. Greenwood, Lebanon, Ill.; G. B. M. Zerr, Parsons, W. Va.; A. H. Holmes, Brunswick, Maine, and the Proposer.

214. Proposed by H. F. MacNEISH, A. B., University High School, Chicago, Ill.

Inscribe in a given circle a triangle whose sides shall pass through three given points.

Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Denote the fixed points by X, Y, Z. Take on the circle any number of points A_1, A_2, ... Draw A_1X cutting the circle again in B_1; draw B_1Y cutting the circle again in C_1; draw C_1Z cutting the circle again in D_1.

Perform the same operation with A_2, ... thus determining the points D_2, ... The ranges A_1, A_2, ..., D_1, D_2, ... are homographic. Construct their double points which will give two (real or imaginary) positions of a point A with which D will coincide.

CALCULUS.

Solutions of Problem 172 have been received from J. E. Sanders, Hackney, Ohio, and J. B. Gregg, M. Sc., C. E., Senecaville, Ohio.

173. Proposed by J. E. SANDERS, Hackney, Ohio.

Find the greatest ellipse that can be inscribed in a quadrant of a given circle.

Solution by J. B. GREGG, M. Sc., C. E., Senecaville, Ohio, and G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

By symmetry, the radius bisecting the quadrant will coincide with an axis of the ellipse. Denote the length of this semi-axis, and that of the transverse semi-axis, by a, b. Then

πab=area of ellipse=A (say).

$a+\sqrt{(a^2+b^2)}$=radius of given circle=r (say).

$\therefore r^2=2ar+b^2$.

Since A is a maximum, $bda+adb=0$. Also $rda+bdb=0$.

$\therefore ra=b^2$, $r=3a$, $A=\frac{1}{6}\pi r^2\sqrt{3}$.

Also solved by G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va., and A. H. Holmes, Brunswick, Me.

MECHANICS.

163. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

A particle A, mass m, rests on a smooth horizontal plane and is attached by two inelastic strings to masses m_1, m_2 at points B and C such that BAC is a right angle. If a blow is given A at an angle θ to AB, find the initial direction of motion of m, and equations for initial motion of the particles m_1 and m_2.

Solution by J. B. GREGG, M. Sc., C. E., Senecaville, Ohio.

Let v, v_1, and v_2 be the respective initial velocities of m, m_1, and m_2, and let ϕ be the angle which the initial motion of m makes with the line of direction of the blow. Construct $AD=v$; then DE perpendicular to BA produced=v_2, $AE=v_1$, $\angle DAE=\theta+\phi$.

$v_1=v\cos(\theta+\phi)$, $v_2=v\sin(\theta+\phi)$, $mv\sin\phi+m_1v_1\sin\theta=m_2v_2\cos\theta$.

Solving for ϕ, $\tan\phi=\dfrac{(m_2-m_1)\sin\theta\cos\theta}{m-m_1\sin^2\theta-m_2\cos^2\theta}$.

DIOPHANTINE ANALYSIS.

116. Proposed by HARRY S. VANDIVER, Bala, Pa.

If n is an odd positive integer, and 1, n, n', n'', ... denote all its distinct divisors, then $2^n>2(n+1)(n'+1)(n''+1)$.....

II. Solution by the PROPOSER.

By Euler's generalization of Fermat's Theorem (Dirichlet-Dedikind *Zah-*

lentheorie, 4th edition, p. 38), we have for n odd,

$$2^{\phi(1)}-1\equiv 0(\mathrm{mod}1),\ 2^{\phi(n)}-1\equiv 0(\mathrm{mod}n),\ 2^{\phi(n')}-1\equiv 0(\mathrm{mod}n'),\ldots\ldots$$

where $\phi(n)$ is the number of integers less than n and prime to it. We have, then, since $\phi(1)=1$,

$$2^{\phi(1)}-1=1,\ 2^{\phi(n)}-1\geq n,\ 2^{\phi(n')}-1\geq n',\ldots\ldots$$

Now, unless $n=3$, there is at least one of the latter relations which is not an equality. For, let p be the largest prime factor of n, then $2^{p-1}-1\geq p$.

Make $p=3$, then $2^{3-1}-1=3$. From this relation it is evident that $2^{p-1}-1>p$ for $p>3$. If $p=3$, then $n=3^k$. But

$$2^{\phi(3^2)}-1>3^2.$$

Hence, unless $k=1$ or $n=3$, there is at least one inequality in the original scheme, and we get by transposition and multiplication,

$$2^{\phi(1)+\phi(n)+\phi(n')\cdots}>2(n+1)(n'+1)(n''+1)\ldots\ldots$$

or, since $\phi(1)+\phi(n)+\phi(n')\ldots=n$ (Dirichlet, l. c., p. 26),

$$2^n>2(n+1)(n'+1)(n''+1)\ldots\ldots$$

for $n>3$, which is the theorem.

118. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Find the two least *positive** integral numbers such that their sum shall be a square and the sum of their squares a biquadrate.

Professor Walker finds that the two numbers,

$$x=4{,}565{,}468{,}027{,}761,\quad y=1{,}061{,}625{,}293{,}520$$

have the properties that their sum is a square and the sum of their sqnares is a biquadrate. Indeed, it may be verified that $(x+y)^{\frac{1}{2}}=2372159$; $(x^2+y^2)^{\frac{1}{4}}=2165017$. In the letter accompanying the solution Professor Walker asks "are these the least numbers?" We invite other solutions bearing upon this important point of the problem.

119. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

If p be any prime number and n any positive integer, the congruence $x^{p^n}\equiv x(\mathrm{mod}p^n)$ has p and only p solutions modp^n. Hence the congruence defines the Galois field of order p^n if and only if $n=1$.

*As originally printed the word "positive" was omitted.

Solution by the PROPOSER.

The only solution x divisible by p is seen to be $\equiv 0 (\mathrm{mod} p^n)$. Let next x be prime to p. Then

$$x^{p^n-1} \equiv 1 (\mathrm{mod} p^n), \quad x^{p^n-2(p-1)} \equiv 1 (\mathrm{mod} p^n),$$

the second from Fermat's theorem generalized, since $\phi(p^n) = p^{n-1}(p-1)$. Hence must $x^{p-1} \equiv 1 (\mathrm{mod} p^n)$. To prove that the latter has precisely $p-1$ distinct solutions $\mathrm{mod} p^n$, we proceed by induction, the result being true for $n=1$ by Fermat's theorem. Let therefore $x_1, \ldots, x_{p-1}$ be the distinct roots of $x^{p-1} \equiv 1 (\mathrm{mod} p^a)$, whence $x_i^p = x_i + r_i p^a$, each r_i being a fixed integer. To determine the roots of $x^{p-1} \equiv 1 (\mathrm{mod} p^{a+1})$, we seek the integral values of m_i incongruent mod$p$ such that $x = x_i + m_i p^a$ satifies $x^p \equiv x (\mathrm{mod} p^{a+1})$. But $(x_i + m_i p^a)^p = x_i^p +$ multiple of p^{a+1}, by the Binomial Theorem. The condition is therefore

$$x_i + r_i p^a \equiv x_i + m_i p^a \ (\mathrm{mod} p^{a+1}),$$

whence $m_i \equiv r_i (\mathrm{mod} p)$ and m_i is uniquely determined modulo p.

AVERAGE AND PROBABILITY.

148. Proposed by M. C. RORTY, Boston, Mass.

Assuming n points to fall at random upon a circle of circumference a, what is the probability of m or more points falling within a length b upon this circumference?

NOTE. This problem has practical application in determining the probability of accidental rushes of telephone calls as distinct from those rushes which are due to commercial causes. The solution for m or more points falling within a specified length b is known. The problem presented above differs from this in that a solution is required for *any* length b.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $m+r$, $(r=0, 1, 2, 3, \ldots n-m)$ be the number of points to fall on b. $m+r$ points can be selected from n points in $n! \div (m+r)!(n-m-r)!$ ways.

$$\therefore \text{Chance} = \frac{2n!}{(m+r)!(n-m-r)!} \cdot \frac{(b)^{m+r}(a-b)^{n-m-r}}{a^n}.$$

149. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Three points are taken at random on the convex surface of a right cone. Find the probability that the section of the cone made by the plane passing through them is a complete ellipse.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be a section of the cone, E one of the random points in the surface. $AD=c$, $GD=h$, $BD=R$, $EG=r$. Through E pass the plane EC. Then the area of the surface, of which EBC is a projection

$$= \frac{\pi \sqrt{[h^2+(R-r)^2]}}{R-r} [R^2 - (R+r)\sqrt{(Rr)}].$$

(See Vol. V, Nos. 6-7). Now $h=(c/R)(R-r)$.

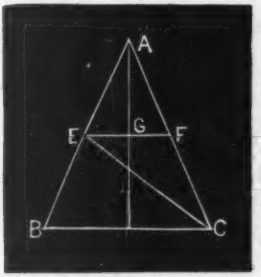
$\therefore$ Surface of which EBC is a projection $=\frac{\pi}{R}\sqrt{(R^2+c^2)}\,[R^2-\frac{1}{2}\sqrt{(Rr)(R+r)}]$.

S=surface of which $EAFC$ is the projection.

$$\therefore S=\pi R\sqrt{(R^2+c^2)}-\frac{\pi}{R}\sqrt{(R^2+c^2)}[R^2-\tfrac{1}{2}\sqrt{(Rr)(R+r)}]$$

$$=\frac{\pi\sqrt{(Rr)(R+r)}\sqrt{(R^2+c^2)}}{2r}.$$

If the two remaining points are taken anywhere on the surface S the section will be a complete ellipse.



$$\therefore \text{The chance}=\frac{\int_0^R \pi^2(R^2+c^2)r(R+r)^2\,dr}{4\pi^2R^3(R^2+c^2)\int_0^R dr}$$

$$=\frac{1}{4R^4}\int_0^R (R^2r+2Rr^2+r^3)dr=\tfrac{17}{48}.$$

150. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If the length of a circular arc be b and the radius vary uniformly, what is the average area of all the segments possible?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The least area is=0 when R=radius is ∞. The greatest area is the area of a circle, $R=b/(2\pi)$, circumference$=b$, $\therefore$ area$=b^2/(4\pi)$.

$$\therefore \text{Average area}=\tfrac{1}{2}\left(\frac{b^2}{4\pi}+0\right)=\frac{b^2}{8\pi}.$$

The problem may also be solved as follows: Area of segment

$=\frac{1}{2}Rb-\frac{1}{2}R^2\sin(b/R)$. Average area$=\triangle=\frac{1}{2}\int_{b/2\pi}^{\infty}(Rb-R^2\sin b/R)dR/\int_{b/2\pi}^{\infty}dR.$

Let $b/R=y$, then limits of y are 0 and 2π.

$$\therefore \triangle=\tfrac{1}{2}b^2\int_0^{2\pi}(1-\frac{1}{y}\sin y)\frac{dy}{y^3}/\int_0^{2\pi}\frac{dy}{y^2}$$

$$=\tfrac{1}{2}b^2\left\{\frac{\left[-\frac{1}{2y^2}+\frac{\sin y}{3y^2}+\frac{\cos y}{6y^2}-\frac{\sin y}{6y}+\frac{1}{6}\left(\log y-\frac{y^2}{2.2!}+\frac{y^4}{4.4!}-\frac{y^6}{6.6!}+\cdots\right)\right]_0^{2\pi}}{\left[-\frac{1}{y}\right]_0^{2\pi}}\right\}$$

$$=b^2/8\pi.$$

151. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

If A and B play tennis together, find the probability that A will win, given that a is the probability that A will win a given point and b the probability that B will win the point.

I. Solution* by the PROPOSER.

Without reaching the score deuce, A wins [a game] only in the following cases:

AAAA, AAABA, AABAA, ABAAA, BAAAA,
AAABBA, AABBAA, ABBAAA, BBAAAA, AABABA,
ABAABA, ABABAA, BAABAA, BABAAA, BAAABA,

with the probability $a^4+4a^4b+10a^4b^2$. The number of ways in which the score deuce may first arise equals the number of permutations of A, A, A, B, B, B, which is $6!\div(3!3!)=20$. It remains to find the probability that A will win after the score deuce is reached. This happens in 2 plays in the case AA; in 4 plays in the cases ABAA, BAAA; in 6 plays in the cases ABABAA, ABBAAA, BAABAA, BABAAA; in 8 plays in the cases when either AB or BA is followed by one of the preceding 4 cases; etc. Hence the probability that A will win after deuce is

$$a^2+2a^3b+4a^4b^2+8a^5b^3+\ldots+2^na^{n+2}b^n+\ldots=\frac{a^2}{1-2ab}.$$

The total probability that A will win is therefore

$$a^4+4a^4b+10a^4b^2+\frac{20a^2}{1-2ab}a^3b^3.$$

Interchanging a and b, we obtain as the probability that B will win

$$b^4+4b^4a+10b^4a^2+\frac{20b^2}{1-2ab}b^3a^3.$$

As a check, it is seen that this sum equals 1 in view of $a+b=1$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

To win a game in the first six points A must win four straight points, or four out of five, or four out of six. The chance of this is

$$a^4+\frac{4a^4b}{1!}+\frac{4.5a^4b^2}{2!}=a^4+4a^4b+10a^4b^2.$$

The chance that B wins in the first six points is $b^4+4ab^4+10a^2b^4$. The chance that each wins three points is $(6!/3!3!)a^3b^3=20a^3b^3$. If they each win three points, then A must win two successive points to win the game. The chance that A wins two successive points is $20a^5b^3$, the chance that B wins two successive points is $20a^3b^5$. The chance that each win a point is $40a^4b^4$. After each have

*The problem was intended to read "win a game," a second proposed problem reading "win a set." The latter has been withdrawn in view of Professor Zerr's solution.

won four points, the chance that A wins two successive points is $40a^6b^4$. The chance that B wins two successive points is $40a^4b^6$, the chance that each win a point is $80a^5b^5$, and so on.

Let p=A's chance of winning the first game, and q=B's chance of winning the first game.

$$\therefore p=a^4+4a^4b+10a^4b^2+20a^5b^3+40a^6b^4+80a^7b^5+160a^8b^6+\ldots\ldots$$
$$=a^4+4a^4b+10a^4b^2+20a^5b^3(1+2ab+4a^2b^2+8a^3b^3+\ldots\ldots)$$
$$=a^4+4a^4b+10a^4b^2+\frac{20a^5b^3}{1-2ab}=\frac{a^4}{1-2ab}(1+4b+10b^2-2ab-8ab^2).$$

Similarly, $q=\frac{b^4}{1-2ab}(1+4a+10a^2-2ab-8a^2b)$.

The chance that A wins a set in ten games is

$$p^6+6p^6q+\frac{6.7}{2!}p^6q^2+\frac{6.7.8}{3!}p^6q^3+\frac{6.7.8.9}{4!}p^6q^4$$
$$=p^6+6p^6q+21p^6q^2+56p^6q^3+126p^6q^4.$$

The chance that B wins the set in ten games is $q^6+6q^6p+21q^6p^2+56q^6p^3+126q^6p^4$.

The chance that each wins five games is $\frac{10!}{5!5!}p^5q^5=252p^5q^5$.

Let P be A's and Q be B's chance of winning the first set. Then as before

$$P=p^6+6p^6q+21p^6q^2+56p^6q^3+126p^6q^4+252p^6q^5+504p^8q^6$$
$$+1008p^9q^7+2016p^{10}q^8+\ldots\ldots$$
$$=p^6+6p^6q+21p^6q^2+56p^6q^3+126p^6q^4+252p^7q^5(1+2pq$$
$$+4p^2q^2+8p^3q^3+\ldots\ldots)$$
$$=p^6+6p^6q+21p^6q^2+56p^6q^3+126p^6q^4+\frac{252\,p^7q^5}{1-2pq}$$
$$=\frac{p^6}{1-2pq}(1+6q+21q^2+56q^3+126q^4-2pq-12pq^2-42pq^3-112pq^4).$$

$$Q=\frac{q^6}{1-2pq}(1+6p+21p^2+56p^3+126p^4-2pq-12p^2q-42p^3q-112p^4q).$$

The chance that A wins three sets out of five and thus wins the match is $P_1=P^3+3P^3Q+6P^3Q^2=P^3(1+3Q+6Q^2)$.

The chance that B wins the match is $Q_1=Q^3(1+3P+6P^2)$.

MISCELLANEOUS.

142. Proposed by R. A. WELLS, Bellevue College, Bellevue, Neb.

Find a general expression for the value of θ such that when θ is one of the acute angles of a right triangle, the three sides of the triangle will be commensurable.

I. Solution by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

Without assuming the triangle to be right, we start with the equation

$$c^2=a^2+b^2-2ab\cos C,\ \therefore a=b\cos C\pm\sqrt{(b^2\cos^2C-b^2+c^2)}.$$

Let $b^2\cos^2 C - b^2 + c^2 = 1^2$. Then $c^2 = 1^2 + b^2\sin^2 C$. Assume $1 = x^2 - y^2$, and $b\sin C = 2xy$. Then $a = \sqrt{(b^2 - 4x^2y^2)} \pm (x^2 - y^2)$.

Let $b^2 - 4x^2y^2 = s^2$, $b + 2xy = ks$, and $b - 2xy = s/k$. Then $b = s(k^2+1)/2k$, $xy = s(k^2-1)/4k$.

Taking $s = 4mk$ we obtain the following formulae:

$$b = 2m(k^2+1), \quad c = x^2 + m^2(k^2-1)^2/x^2,$$
$$a = 4mk \pm [x^2 - m^2(k^2-1)^2/x^2], \quad C = \cos^{-1}[2k/(k^2+1)].$$

As an example, assume $x=3$, $m=3$, $k=2$. We find $a=24$, $b=30$, $c=18$, $\cos C = \frac{4}{5}$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

m^2+n^2, m^2-n^2, $2mn$ are the sides of any right triangle. Let $m = pn$.

$\therefore n^2(p^2+1)$, $n^2(p^2-1)$, $2n^2p$ are the sides.

The equation for the general value of θ is

$$n^2(p^2-1)\cos\theta + 2n^2p\sin\theta = n^2(p^2+1).$$

Let $\sin\beta = \dfrac{n^2(p^2-1)}{n^2(p^2+1)}$, $\cos\beta = \dfrac{2n^2p}{n^2(p^2+1)}$, then the equation becomes

$$\cos\beta\cos\theta + \sin\beta\sin\theta = 1, \quad \cos(\theta - \beta) = 1.$$

$\therefore \theta - \beta = 2\pi n_1$, $\theta = 2\pi n_1 + \beta$.

As θ must be less than $\frac{1}{2}\pi$, the general value of θ is $\theta = \beta$.

Also solved by J. B. Greggs, M. Sc., C. E., Senecaville, Ohio.

PROBLEMS FOR SOLUTION.

ALGEBRA.

192. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, O.

What is the difference between the squares of the two *infinite* continued fractions $\left(3 + \dfrac{1}{6+\text{etc.}}\right)$ and $\left(2 + \dfrac{1}{4+\text{etc.}}\right)$?

193. Proposed by SAUL EPSTEEN, Ph. D., Chicago, Ill.

Professor Goursat states (*Transactions of the American Mathematical Society*, January, 1904, p. 111) that if $a_1, a_2, \ldots, a_n$; $h_1, h_2, \ldots, h_n$ are two sequences, the h's being all positive, then $\sum \dfrac{a_i^2}{h_i^2} \geq \dfrac{(\Sigma a_i)^2}{\Sigma h_i}$. Prove this.

194. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In the determination of the canonical forms of Abelian transformations modulo p, one is led to the type $[b_1, b_2, b_3]$:

$$\xi_1' = \xi_1,\ \eta_1' = b_1\xi_1 + \eta_1 + b_2\xi_2 + \eta_2 + b_3\xi_3 + \eta_3,\ \xi_2' = \xi_2 - \xi_1,$$
$$\eta_2' = \eta_2 + b_2\xi_2,\ \xi_3' = \xi_3 - \xi_1,\ \eta_3' = \eta_3 + b_3\xi_3.$$

Find its period and determine the conditions under which it is conjugate with $[c_1, c_2, c_3]$ under Abelian transformation.

GEOMETRY.

219. Proposed by H. F. MacNEISH, A.B., Assistant in Mathematics, University High School, Chicago, Ill.

Draw a line through a given point which shall divide a given quadrilateral into two equivalent parts; (1) when the point lies in a side of the quadrilateral, (2) when the point is without, (3) within the quadrilateral.

220. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, West Va.

Two triangles are circumscribed to a given triangle ABC, having their sides perpendicular to the sides of the given triangle. Prove that the two triangles are equal, and find the area of these triangles.

DIOPHANTINE ANALYSIS.

120. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Find the prime numbers p for which $x^2 - pxz - px - z + p^2 - 3 = 0$ has more than two sets of positive integral solutions x, z, each $< p$.

AVERAGE AND PROBABILITY.

152. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

A square hole is cut through the center of a sphere of radius r. Show that the average volume removed is $\frac{1}{12}r^3(23\sqrt{2} - 28)$.

MISCELLANEOUS.

143. Proposed by R. L. MOORE, A. M., Fellow in Mathematics, The University of Chicago.

Does there exist a function, of single valued inverse, satisfying the following conditions: (1) if $u = f(v)$, $w = f(u)$ then $w = f(v)$; (2) given $u_1 = f(u)$, $v_1 = f(v)$, $w_1 = f(w)$; if the triple w, v, w can be regarded as the lengths of the sides of a triangle, the triple u_1, v_1, w_1 cannot be similarly regarded.

144. Proposed by IRA M. DeLONG, Professor of Mathematics in the University of Colorado, Boulder, Col.

Determine the number of distinct spherical polygons of n sides formed by arcs of n given great circles on a sphere, each arc to be greater than 0 degrees, and less than 360 degrees, and no two sides of any polygon to lie an the same circle.

NOTES.

Errata in solution of Algebra, 188, p. 14. Read $y=a$, $(1\mp\sqrt{3})a$.

Professor Arnold Emch has been appointed editor of the *University of Colorado Studies* in place of the late Dr. Arthur Allin.

On January 22 occurred the death of the Rev. Dr. George Salmon, the well known and distinguished author of a series of classic mathematical works embracing A Treatise on Conic Sections; Lessons Introductory to the Modern Higher Algebra; A Treatise on Higher Plane Curves; and A Treatise on the Geometry of Three Dimensions. F.

On December 12, 1903, occurred the death of our valued friend and contributor, Marcus Baker, Cartographer of the U. S. Geological Survey and Assistant Secretary of the Carnegie Institution. Mr. Baker was an enthusiastic supporter of the MONTHLY from its beginning. His contributions in the way of excellent solutions will be greatly missed by many of our readers. F.

Beginning with January, 1904, the *Mathematical Supplement of School Science* appears as an independent magazine entitled *School Mathematics* under the responsible editorship of Professor G. W. Myers of the School of Education of The University of Chicago. It is devoted mainly to questions concerning the pedagogy of mathematics, covering the period from the later grammar grades to the end of the sophomore year in college.

The following courses in Advanced Mathematics will be offered in the Summer School of the University of Pennsylvania, opening July 5, 1904: Calculus, Dr. G. H. Hallett; Differential Equations, Dr. F. H. Safford; Modern Analytical Geometry, Head Professor Crawley; Theory of Functions of a Real Variable, Assistant Professor I. J. Schwatt; Theory of Functions of a Complex Variable, Assistant Professor G. E. Fisher. Other courses will be offered by Dr. Easton and Mr. Dimick. F.

"Geometric Construction of the Regular Decagon and Pentagon Inscribed in a Circle" is the title of a pamphlet just issued by the author, Major Henry H. Ludlow, Professor of Military Science at Agricultural College, Mississippi. The principal construction is that proposed in the MONTHLY, November, 1903 (Geometry 211). Introductory paragraphs furnish the basis of a strictly elementary proof, independent of the results following from the usual construction. The Preface gives an interesting bit of history in regard to the construction. D.

The University of Chicago announces the following appointments of graduates and graduate students to mathematical positions of collegiate rank: Otto F. Geckeley, Instructor in Mathematics at the School of Technology, Atlanta, Ga.; Monro N. Work, Instructor in Mathematics at the Georgia State Industrial College; Theodore Lindquist, Instructor in Mathematics at Augustana College, Rock Island, Ill.; A. M. Salsbury, Instructor in Mathematics at the Illinois College, Jacksonville, Ill.; Duane Studley, Instructor in Mathematics at the Lewis Institute, Chicago, Ill.; F. W. Hanawalt, Professor of Mathematics in Albion College, Albion, Mich.

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SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE III. THE CONCEPT OF ORDER; ANGLES AND TRIANGLES.

Theorem 6. *All the lines which pass through the same point O pass also through a common point O'.*

Let the two lines a and b which pass through the point O meet again in some point O'. Let c be any other line which passes through the point O. Then c must cut each of the lines a and b in another point—say Q and R respectively. The pairs of points O and O', O and Q, O and R do not uniquely determine a line: for we have by the above hypothesis two lines through each pair of points. Now Axiom I states that there exists only one point which taken with the point O will not uniquely determine a line. Hence the points O' and Q and R must coincide and the theorem is proved.

Theorem 7. *If in a motion one point of the surface remains stationary, that other point which taken with it will not uniquely determine a line remains also stationary.*

If O be the point which remains fixed, any direction a issuing from O will be carried into a direction a' which issues from O. Hence if b and c are any two lines passing through O they will be carried into two lines b' and c' which likewise pass through O. But by the foregoing theorem b and c intersect in O' and b' and c' also intersect in O'. Hence O' must be carried into O'—that is, it remains unchanged in position.

Two points such as O and O' through which an indefinitely great number of lines may be passed are called *antipodal* points. The segment of a line intercepted between the two points is called a *semi-line*.